VECTOR CALCULUS :

Lecture - 1 :

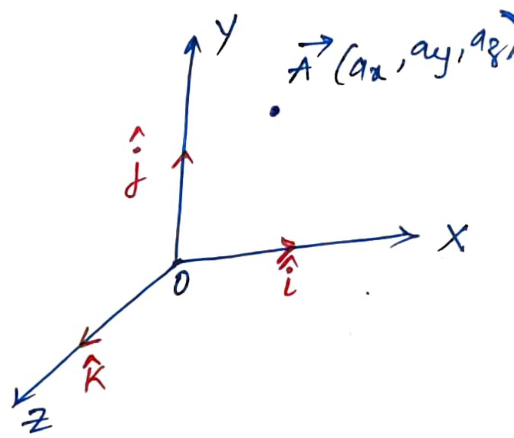
BASICS OF VECTORS :

Vector :

It is a quantity which has both magnitude & direction.

$$\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

Magnitude of $\vec{A} = |\vec{A}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$



Unit vector:

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Product Of Vectors :

There are three types of products.

(i) General Product

(ii) Dot Product (or Inner Product (or Scalar Product.

(iii) Cross Product.

The explainations of the three types of products are given in next page

(i) General Product :

If we want the product of a scalar and a vector which results in no change in direction of the resulting product is called as general product.

Suppose $\vec{A}$ is a vector quantity

$$\vec{A} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

& 'l' as a scalar quantity.

Now General product of $\vec{A}$ & l $= (\vec{A} \cdot l) = \vec{B}$

$$(\vec{A} \cdot l) = l(a_x \hat{i} + a_y \hat{j} + a_z \hat{k})$$

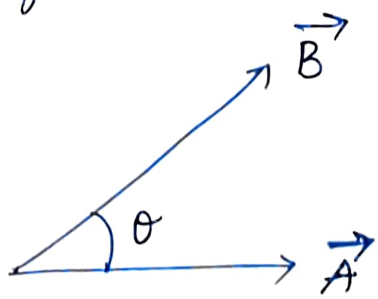
$$\boxed{\vec{B} = (a_x l)\hat{i} + (l a_y)\hat{j} + (l a_z)\hat{k}.}$$

(ii) Dot Product (OR) Inner Product (OR) Scalar Product :

Dot product is performed b/w two vectors.

Dot product of $\vec{A}$ & $\vec{B}$ is given as

$$\boxed{\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta}$$



$$\boxed{\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}\right)}$$

Mathematically:

$$\vec{A}\cdot\vec{B} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\vec{A}\cdot\vec{B} = (a_x b_x + a_y b_y + a_z b_z)$$

Here the dot product of two vectors $\vec{A}$ & $\vec{B}$ gives a scalar quantity. That is the reason we call the ~~scalar~~ dot product of $\vec{A}$ & $\vec{B}$ as scalar product.

(iii) CROSS PRODUCT :

The cross product of two vector $\vec{A}$ & $\vec{B}$ is given as

$$\boxed{\vec{A}\times\vec{B} = |\vec{A}|\,|\vec{B}|\sin\theta\ \hat{n}}$$

where $\hat{n}$ is unit normal vector ⊥er to both vectors $\vec{A}$ & $\vec{B}$.

Mathematically:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

~~The vector product~~

The cross product of $\vec{A}$ & $\vec{B}$ vectors gives us the vector quantity. That is the reason we call cross product as vector product.

Eg: Given $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$ & $\vec{B} = \hat{i} + 2\hat{j}$

$$\vec{A} \cdot \vec{B} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j}) = 2 - 2 = 0$$

$$\vec{A} \cdot \vec{B} = 0 \Rightarrow \vec{A} \perp \vec{B}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = \hat{i}(-2) - \hat{j}(-1) + \hat{k}(5)$$

$$\vec{A} \times \vec{B} = -\cancel{2\hat{j}+\hat{j}}\ -2\hat{i} + \hat{j} + 5\hat{k}$$

Orthogonal Vectors :

Two vector are said to be orthogonal if their dot product is '0' i.e, angle b/w them is 90°.

$$\vec{A} \cdot \vec{B} = 0$$

$$|\vec{A}|\,|\vec{B}| \cos\theta = 0$$

$$\cos\theta = 0$$

$$\boxed{\theta = 90^\circ}$$

2. Orthonormal Vectors :

Two vectors are said to be orthonormal if both of their magnitudes is 1 & their dot product must be '0'.

Consider two vectors as $\vec{A}$ & $\vec{B}$.

If $\left.\begin{array}{l}\vec{A}\cdot\vec{B}=0\\ |\vec{A}|=|\vec{B}|=1\end{array}\right\}$ then $\vec{A}$ & $\vec{B}$ are said to be orthonormal.

3. Collinear Vectors :

Two vectors are said to be collinear if the angle b/w them is 0°.

$$|\vec{A}|\cdot|\vec{B}|\cos 0^\circ = \vec{A}\cdot\vec{B}$$

If $\boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|}$ then $\vec{A}$ & $\vec{B}$ are said to be collinear vectors.

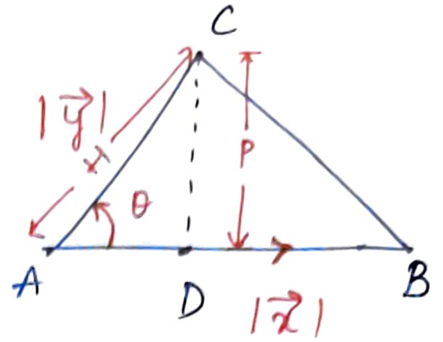
4. Parallel Vectors :

Two vectors are said to be parallel vectors if both of the vectors are proportional to each other

If $\boxed{\vec{B} = K\vec{A}}$ where K = constant.

then $\vec{A}$ & $\vec{B}$ are said to be parallel vectors.

5. Area Of Triangle :



Area of the given Δ^{le} ABC $= \frac{1}{2} \overline{AB} \cdot \overline{CD}$

$= \frac{1}{2} |\vec{x}| |\vec{y}| \sin\theta$

$\sin\theta = \frac{P}{H}$

$P = H \sin\theta$.

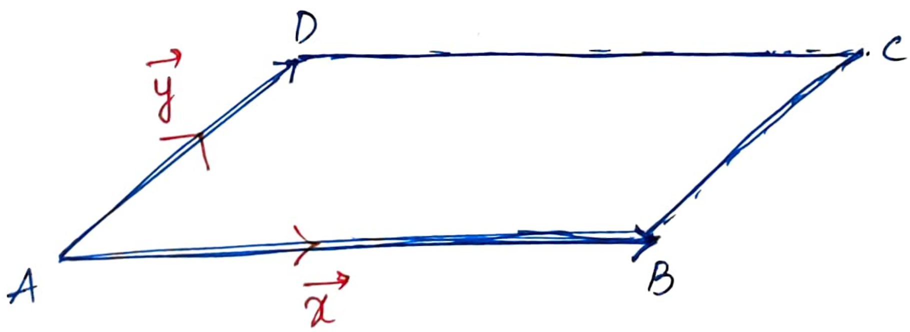
$$\boxed{\therefore \text{Area of } \Delta^{le} \text{ ABC} = \frac{1}{2} |\vec{x} \times \vec{y}|}$$

Area of Δ^{le} ABC $= \frac{1}{2} \overline{AB} \cdot \overline{CD}$

$= \frac{1}{2} |\vec{x}| |\vec{y}| \sin\theta$

$$\boxed{\therefore \text{Area of } \Delta^{le} \text{ ABC} = \frac{1}{2} |\vec{x} \times \vec{y}|}$$

6. Area of Parallelogram :



$$\boxed{\text{Area of parallelogram ABCD} = |\vec{x} \times \vec{y}|}$$

WORKBOOK QUESTIONS :

OBJECTIVE TYPE QUESTIONS :

Q2. The two vectors $[1, 1, 1]$ and $[1, a, a^2]$ where $a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, are

A) Orthonormal

B) Orthogonal

C) Parallel

D) Collinear.

Soln: Let $\vec{A} = \hat{i} + \hat{j} + \hat{k}$

$\vec{B} = \hat{i} + a\hat{j} + a^2\hat{k}$.

$|\vec{A}| = \sqrt{3}$; $|\vec{B}| = \sqrt{1 + a^2 + a^4} = \sqrt{1 + w^2 + w} = 0$

$|\vec{A}| = \sqrt{3}$ & $|\vec{B}| = 0$

Clearly for orthonormal : $|\vec{A}| = |\vec{B}| = 1$ &

$\vec{A} \cdot \vec{B} = 0$

So $\vec{A}$ & $\vec{B}$ are not orthonormal.

$\vec{A} \cdot \vec{B} = (\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} + a\hat{j} + a^2\hat{k})$

$= 1 + a + a^2$

$= 1 + w + w^2$ ($\because$ Cube root of unity)

$$\boxed{\vec{A} \cdot \vec{B} = 0}$$

So $\vec{A}$ & $\vec{B}$ are orthogonal vectors. $\boxed{\text{Ans: (B)}}$

Q3. If $\vec{a}$ and $\vec{b}$ are two arbitrary vectors with magnitudes a and b, respectively, $|\vec{a} \times \vec{b}|^2$ will be equal to

A) $a^2b^2 - (\vec{a}\cdot\vec{b})^2$ B) $ab - \vec{a}\cdot\vec{b}$

C) $a^2b^2 + (\vec{a}\cdot\vec{b})^2$ D) $ab + \vec{a}\cdot\vec{b}$

Soln:

$$|\vec{a}\times\vec{b}|^2 = \left||\vec{a}||\vec{b}| \sin\theta\, \hat{n}\right|^2$$

$$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$= a^2b^2 \sin^2\theta$$

$$= a^2b^2 (1 - \cos^2\theta)$$

$$= a^2b^2 - a^2b^2\cos^2\theta$$

$$\boxed{\therefore |\vec{a}\times\vec{b}|^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2}$$

$\boxed{\text{Ans: (A)}}$

Q4) For the parallelogram OPQR shown in the sketch $\vec{OP} = a\hat{i} + b\hat{j}$ and $\vec{OR} = c\hat{i} + d\hat{j}$. The area of the parallelogram is

A) $ad - bc$ B) $ac + bd$

C) $ad + bc$ D) $ab - cd$

Soln: We have

Area of Parallelogram OPQR $= |\vec{OP} \times \vec{OR}|$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \hat{k}(ad - bc)$$

$$\boxed{\therefore \text{ Area of } ||^{gm} \text{ OPQR} = |(ad-bc)\hat{k}| = ad - bc}$$

Ans: (A)

Q5) If $A(0,4,3)$ $B(0,0,0)$ and $C(3,0,4)$ are three points defined in x, y, z co-ordinate system then which one of the following vectors is ⊥er to both the line vectors $\vec{BA}$ and $\vec{BC}$?

A) $16\hat{i} + 9\hat{j} - 12\hat{k}$ B) $16\hat{i} - 9\hat{j} + 12\hat{k}$

C) $16\hat{i} - 9\hat{j} - 12\hat{k}$ D) $16\hat{i} + 9\hat{j} + 12\hat{k}$

Soln: $\vec{BA} = \vec{OA} - \vec{OB} = 4\hat{j} + 3\hat{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 3\hat{i} + 4\hat{k}$

$$(\vec{BA} \times \vec{BC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & 3 \\ 3 & 0 & 4 \end{vmatrix} = \hat{i}(16) - \hat{j}(-9) + \hat{k}(-12)$$

$$(\vec{BA} \times \vec{BC}) = 16\hat{i} + 9\hat{j} - 12\hat{k}$$

Ans : (A)

VECTOR CALCULUS

Lecture - 2 :

DEL OPERATOR

Vector Differential Calculus :

We have very important vector differential operator. It is called as "Del". It is denoted as '∇'

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

Units of ∇ : $\frac{1}{m}$ (or) m^{-1}

Operations of Del Operator :

1. General Product $\rightarrow$ $\nabla[\phi(x, y, z)]$ $\rightarrow$ grad(ϕ), Gradient

2. Dot Product $\rightarrow$ $\nabla \cdot \vec{A}$ $\rightarrow$ Divergence

3. Cross Product $\rightarrow$ $\nabla \times \vec{A}$ $\rightarrow$ Curl

Tip To Remember :

1. General product $\rightarrow$ Gradient

2. Dot Product $\rightarrow$ Divergence

3. Cross Product $\rightarrow$ Curl

Lecture - 03 :

GRADIENT , DIVERGENCE & CURL & DIRECTIONAL DERIVATIVE

GRADIENT : (Gradient → General product)

① "GRADIENT" is only valid for "scalar point function".

"Scalar point functn" means a quantity which has no direction

$$\phi(x,y,z) = x^2 z + xyz + \frac{x}{z^2 y^3}$$

② It is the general operation of del operator to any scalar point function at any point.

$$\boxed{grad(\phi) = \nabla(\phi)}$$

③. Resultant of gradient is a vector quantity.

④. Here the result i.e. vector quantity is referred as "normal vector".

$$\boxed{\nabla(\phi) = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}}$$

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

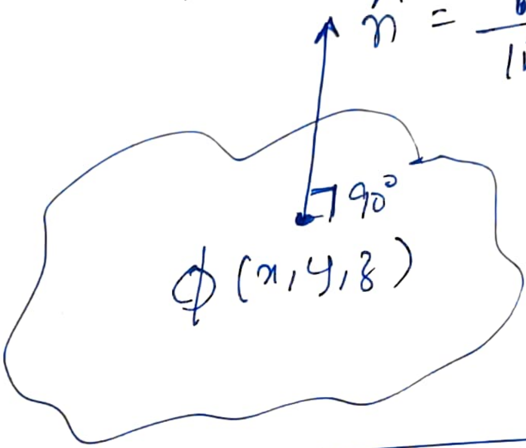
⑤ Here '∇' (del operator) is a vector which is applied on a scalar quantity generally.

PHYSICAL SIGNIFICANCE OF GRADIENT :

1. "GRADIENT" gives the "maximum rate of change".

2. For any arbitary scalar point function, with the help of applying gradient of scalar point functn we can find the normal vector as a result. also we can ~~fand~~ find the direction of normal vector by dividing with its magnitude.

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\text{grad } \phi}{|\text{grad } \phi|}$$



i.e,
$$\boxed{\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\text{grad } \phi}{|\text{grad } \phi|}}$$

Example:
Given $\phi(x,y,z) = xy + yz$. find the grad(ϕ) at $(1,1,1)$?

soln: $\phi(x,y,z) = xy + yz$

grad $\phi(x,y,z) = \nabla(\phi)$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

(0)

positive (+ve)

$$\text{grad}(\phi) = \hat{i}\frac{\partial}{\partial x}(xy+yz) + \hat{j}\frac{\partial}{\partial y}(xy+yz) + \hat{k}\frac{\partial}{\partial z}(xy+yz)$$

$$\text{grad}(\phi) = \hat{i}[y] + \hat{j}[x+z] + \hat{k}[y]$$

$$\left[\text{grad}(\phi)\right]_{at\,(1,1,1)} = \hat{i} + 2\hat{j} + \hat{k} = \text{normal vector of } \phi(x,y,z)$$

$$\boxed{\therefore\ \text{grad}(\phi) \text{ at } (1,1,1) = \hat{i} + 2\hat{j} + \hat{k}}$$

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(5) Here '∇' (del op

is applied on a s