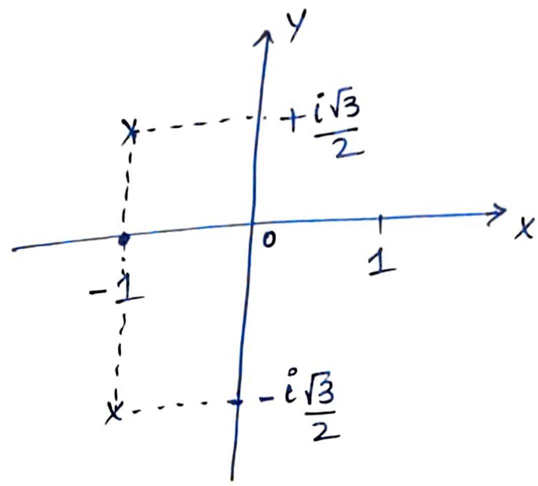
CHAPTER : COMPLEX VARIABLES

Lecture - 1

BASICS OF COMPLEX VARIABLES :

→ Always the complex number will be generated with the conjugate.



i.e $z = x + iy$

$\bar{z} = x - iy$

NOTE:

1. $\begin{cases} i = \sqrt{-1} \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{cases}$

2. $i + i^2 + i^3 + i^4 = 0$: consecutive first four.

3. $i + i^3 + i^5 + i^7 = 0$; odd.

4. $i^2 + i^4 + i^6 + i^8 = 0$: even.

Complex Number:

$z = x + iy$;

Here z is called complex number.

$$\boxed{f(z) = f(x+iy)}$$

Here z is called complex variable.

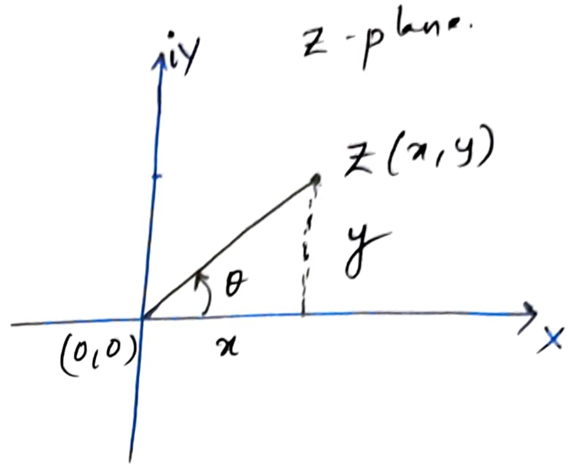
<u>Basic:</u>

$$z = x+iy$$

$$\boxed{\text{Modulus}: |z| = \sqrt{x^2+y^2}}$$

Angle (Arg)(z) $= \theta$

$$\boxed{\text{Arg}(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)}$$



<u>Polar form:</u>

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2+y^2 = r^2$$

$$\boxed{r = |z| = \sqrt{x^2+y^2}}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = x+iy = r\cos\theta + ir\sin\theta$$

$$z = r\left[\cos\theta + i\sin\theta\right]$$

$$\boxed{z = re^{i\theta}} \qquad \boxed{|z| = |r|\,|e^{i\theta}|} \qquad |e^{i\theta}| = 1$$

$$\boxed{|z| = |r| = \sqrt{x^2+y^2}}$$

<u>Basics:</u>

$$z = x + iy$$

$$\bar{z} = x - iy$$

1. $z + \bar{z} = 2x$ (Real)

2. $z - \bar{z} = 2iy = i(2y)$ (purely imaginary)

3. $z\bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$

$$\boxed{\therefore\ z\bar{z} = |z|^2}$$

(*) $\dfrac{z}{\bar{z}} = \dfrac{x+iy}{x-iy}$

Sol^n $\dfrac{z}{\bar{z}} = \dfrac{x+iy}{x-iy} \times \dfrac{x+iy}{x+iy} = \dfrac{(x+iy)^2}{x^2+y^2} = \dfrac{(x^2-y^2) + i\,2xy}{(x^2+y^2)}$

$$\frac{z}{\bar{z}} = \left(\frac{x^2-y^2}{x^2+y^2}\right) + i\left(\frac{2xy}{x^2+y^2}\right) = u + iv$$

$$\therefore\ u = \frac{x^2-y^2}{x^2+y^2}\ ; \quad v = \frac{2xy}{x^2+y^2}.$$

(*) $f(z) = e^z$

soln. $f(z) = e^{x+iy} = e^x \cdot e^{iy} = e^x[\cos y + i \sin y]$

$f(z) = (e^x \cos y) + i(e^x \sin y) = u + iv$

$u = e^x \cos y$ & $v = e^x \sin y$.

(*) $f(z) = \sin z$

soln $f(z) = \sin(x+iy)$

$f(z) = \sin x \cos iy + \cos x \sin(iy)$.

$f(z) = \sin x \cosh \theta + i \cos x \sinh \theta$

$f(z) = u + iv$

$\therefore$ $u = \sin x \cosh \theta$; $v = \cos x \sinh \theta$.

NOTE:

$\cos i\theta = \cosh \theta$

$\sin i\theta = i \sinh \theta$

(*) $f(z) = \ln z$

soln. $f(z) = \ln(x+iy) = u + iv$

$x = r\cos\theta$; $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$f(z) = \ln[r\cos\theta + ir\sin\theta] = \ln r[\cos\theta + i\sin\theta]$

$= \ln r \cdot e^{i\theta} = \ln r + \ln e^{i\theta}$

$$= \ln\sqrt{x^2+y^2} + i\theta$$

$$f(z) = \left[\frac{1}{2}\ln(x^2+y^2)\right] + i\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$f(z) = u + iv$$

$$\therefore\ u = \frac{1}{2}\ln(x^2+y^2) \quad \& \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Lecture - 2 :

CONCEPT OF ANALYTIC FUNCTION :

Condition for a complex function to be Analytic :

The complex function should be

(i) Differentiable

(ii) It should satisfy C-R (Cauchy Riemann) Eqns.

Now

$$f(z) = u(x, y) + i\, v(x, y)$$

If at any point say $z = a$, the function is not analytic then that point is called as "singular point."

Cauchy Rieman (C-R) Equations :

Consider function $f(x) = u(x, y) + i\, v(x, y)$.

(i) $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \quad \text{(OR)} \quad u_x = v_y}$

&

(ii) $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \quad \text{(or)} \quad u_y = -v_x}$

Q). Chek for the following functions whether they are analytic or not?

a) $f(z) = z^3$

b) $f(z) = |z| z$

c) $f(z) = \cos(z)$.

Soln: a) $f(z) = z^3$:

$$f(z) = (x+iy)^3$$

$$f(z) = x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2$$

$$= x^3 - iy^3 + i\,3x^2y - 3xy^2$$

$$f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv$$

$$\therefore \quad u(x,y) = x^3 - 3xy^2 \quad \& \quad v(x,y) = 3x^2y - y^3.$$

~~lest~~ lets check for C-R eqns :

$u_x = 3x^2 - 3y^2$ $\qquad \boxed{u_x = v_y}$ ✓

$v_y = 3x^2 - 3y^2$

$u_y = -6xy$ $\qquad \boxed{u_y = -v_x}$ ✓

$v_x = 6xy$

$$\boxed{\therefore \; f(z) = z^3 \text{ is analytic}}$$